ON THE STABILITY OF STEADY CONVECTIVE MOTION GENERATED BY INTERNAL HEAT SOURCES

PMM Vol. 34, №4, 1970, pp. 700-705
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(Received February 10, 1970)

Plane-parallel convective motion between vertical planes due to heat sources distributed uniformly in the fluid is considered.

Internal heating in a channel closed top and bottom gives rise to closed convective motion with velocity and temperature profiles which are even with respect to the channel axis.

The investigation is carried out under the assumption that the effect of thermal perturbations is negligible. The spectra of the decrements of the normal hydrodynamic perturbations are determined. It is shown that the motion becomes unstable with respect to perturbations in the form of eddies which arise at the boundaries of convective countercurrents. The neutral curves for the two lower instability modes differing in the relative disposition of eddy chains are constructed.

1. Steady motion. Let us consider a plane vertical layer of thickness $2h$ bounded by the parallel planes $x = \pm h$. The temperature of the two planes are maintained constant and equal (in the discussion to follow this constant temperature serves as the reference point). Internal heat sources of volume density Q are distributed uniformly through the volume. Let us write out the equations of convection with allowance for the internal heat sources,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} + g\beta T\boldsymbol{\gamma} \tag{1.1}$$

$$\frac{\partial T}{\partial t} + \mathbf{v}\nabla T = \chi\Delta T + \frac{Q}{\rho c_p}, \qquad \operatorname{div}\mathbf{v} = 0 \tag{1.2}$$

where all the symbols have their standard meaning.

If the vertical dimension of the channel is sufficiently large, the steady motion in its middle portion can be regarded as plane-parallel (the disposition of the coordinate axes is shown in Fig. 1),

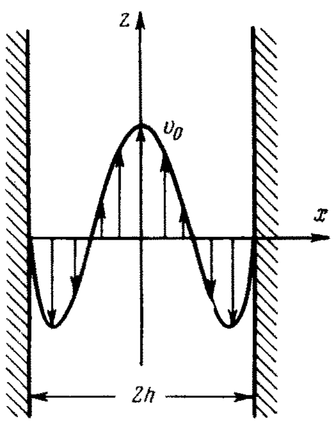

Fig. 1

$$v_x = v_y = 0, \quad v_z = v_0(x), \qquad T = T_0(x), \quad p = p_0(z) \tag{1.3}$$

Equations (1.1) and (1.2) yield the following equations for v_0, T_0 and p_0:

$$\nu v_0'' + g\beta T_0 = \frac{1}{\rho}\frac{dp_0}{dz} = C,$$

$$T_0'' = -q \qquad \left(q = \frac{Q}{\rho c_p \chi} = \text{const}\right) \tag{1.4}$$

Here C is the separation-of-variables constant.

The velocity and temperature vanish at the solid isothermal boundaries of the channel. In addition, we assume that the channel is closed top and bottom; this implies that the discharge over any cross section is equal to zero,

$$v_0(\pm h) = 0, \qquad T_0(\pm h) = 0, \qquad \int_{-h}^{h} v_0 dx = 0 \tag{1.5}$$

From (1.4) and (1.5) we obtain the velocity, temperature, and pressure distributions in plane-parallel steady motion,

$$v_0 = \frac{g\beta q h^4}{120\nu}\left[1 - 6\left(\frac{x}{h}\right)^2 + 5\left(\frac{x}{h}\right)^4\right], \qquad T_0 = \frac{qh^2}{2}\left[1 - \left(\frac{x}{h}\right)^2\right], \qquad \frac{dp_0}{dz} = \frac{2}{5}\rho g\beta q h^2 \tag{1.6}$$

Thus, in contrast to convective flow between planes at different temperatures, the flow in this case has even velocity and temperature profiles. The velocity distribution is shown in Fig. 1. We see that the flow consists of three convective streams, an ascending central stream and two descending streams at the walls. The maximum velocity at the axis is

$$v_m = g\beta q h^4 / 120\nu$$

2. Normal perturbations. We propose to investigate the stability of steady motion (1.6) by the method of perturbations. Let us consider the perturbed motion $\mathbf{v}_0 + \mathbf{v}$, $T_0 + T$, $p_0 + p$, where $\mathbf{v}$, T, p are small unsteady perturbations. In the approximation linear with respect to perturbations, we obtain the following dimensionless perturbation equations from (1.1), (1.2):

$$\partial \mathbf{v}/\partial t + G\left[(\mathbf{v}\nabla)\mathbf{v}_0 + (\mathbf{v}_0\nabla)\mathbf{v}\right] = -\nabla p + \Delta \mathbf{v} + T\boldsymbol{\gamma} \tag{2.1}$$

$$\partial T/\partial t + G\left[\mathbf{v}\nabla T_0 + \mathbf{v}_0 \nabla T\right] = P^{-1}\Delta T, \qquad \operatorname{div}\mathbf{v} = 0 \tag{2.2}$$

$$(G = g\beta q h^5 / 2\nu^2, \; P = \nu/\chi)$$

The units of distance, time, velocity, temperature, and pressure in these expressions are, respectively, h, h^2/ν, $g\beta q h^4 / 2\nu$, $qh^2/2$, $\rho g \beta q h^3 / 2$

The Prandtl number is defined as usual. The Grasshof number is defined in terms of the strength q of the internal heat sources; it is clear that the Grasshof number contains the maximum temperature at the channel axis $^1/_2\, qh^2$ as the characteristic temperature difference. The dimensionless velocity and temperature profiles are of the form

$$v_0 = {}^1/_{60}(1 - 6x^2 + 5x^4), \qquad T_0 = 1 - x^2 \tag{2.3}$$

Let us consider the plane perturbations

$$v_x(x,z,t) = -\partial\psi/\partial z, \; v_y = 0, \; v_y(x, z, t) = \partial\psi/\partial x, \; T = T(x, z, t) \tag{2.4}$$

Here $\psi(x, z, t)$ is the stream function. We set

$$\psi = \varphi(x)\exp(-\lambda t + ikz), \qquad T = \theta(x)\exp(-\lambda t + ikz) \tag{2.5}$$

where φ and θ are the amplitudes of the normal perturbations, λ is the decrement, and k is the wavenumber.

Substituting (2.5) into (2.1) and (2.2), we obtain the amplitude equations

$$\Delta^2\varphi - ikGH\varphi + \theta' = -\lambda\Delta\varphi \tag{2.6}$$

$$P^{-1}\Delta\theta + ikG(T_0'\varphi - v_0\theta) = -\lambda\theta \tag{2.7}$$

$$(\Delta\varphi \equiv \varphi'' - k^2\varphi, \quad H\varphi \equiv v_0\Delta\varphi - v_0''\varphi)$$

The condition whereby the velocity and temperature perturbations vanish at the boundaries of the layer implies the following homogeneous boundary conditions for the amplitudes:

$$\varphi = \varphi' = 0, \ \theta = 0 \qquad \text{for} \quad x = \pm 1 \tag{2.8}$$

Boundary value problem (2.6) – (2.8) defines the spectrum of the characteristic perturbations and their decrements λ.

The results of [1 – 5] imply that in the case of a vertical channel the convective countercurrents are hydrodynamically unstable. The role of thermal perturbations in this case is relatively slight; this is reflected, for example, in the weak dependence of the critical Grasshof number on the Prandtl number. The flow in question also consists of convective countercurrents, which suggests that its crisis (at least for not excessively large Prandtl numbers) is associated with a hydrodynamic mechanism. The stability of this flow can therefore be established in a purely hydrodynamic formulation, i.e. without allowance for thermal perturbations and their effect on the development of hydromechanical perturbations. In this approximation it is necessary to neglect the term containing the lifting force θ' in equation of motion (2.6) and to disregard heat conduction equation (2.7). This reduces analysis to the solution of the Orr-Sommerfeld boundary value problem

$$\Delta^2\varphi - ikGH\varphi = -\lambda\Delta\varphi, \qquad \varphi = \varphi' = 0 \quad \text{for} \quad x = \pm 1 \tag{2.9}$$

with a given velocity profile v_0.

We shall solve the problem by the Bubnov-Galerkin method. As our basic system of functions we take the amplitudes $\varphi_i^{(0)}$ of the perturbations in the quiescent fluid as defined by the boundary value problem

$$\Delta^2\varphi_i^{(0)} = -\mu_i\Delta\varphi_i^{(0)}, \quad \varphi_i^{(0)} = \varphi_i^{(0)\prime} = 0 \quad \text{for} \ x = \pm 1 \quad (i = 0, 1, 2 \ldots) \tag{2.10}$$

(the explicit form of the basis functions is given, for example, in [6]).

Substituting the approximation

$$\varphi = a_0\varphi_0^{(0)} + a_1\varphi_1^{(0)} + \ldots + a_N\varphi_N^{(0)} \tag{2.11}$$

into Eq. (2.9), multiplying by $\varphi_i^{(0)}$ and integrating from minus unity to plus unity over x, we obtain the system of linear homogeneous equations

$$\sum_{n=0}^{N} [(\mu_n - \lambda)\,\delta_{mn} + ikGH_{mn}]\,a_n = 0 \qquad (m = 0, 1, \ldots, N) \tag{2.12}$$

$$H_{mn} = \frac{1}{J_n}\int_{-1}^{1} \varphi_m^{(0)} H\varphi_n^{(0)}\,dx, \qquad J_n = \int_{-1}^{1} \varphi_n^{(0)}\Delta\varphi_n^{(0)}\,dx$$

The characteristic decrements λ are defined as the eigenvalues of the matrix of system (2.12).

Because of the evenness of the velocity profile v_0, boundary value problem (2.9) has two types of solutions, namely solutions which are even and odd with respect to x. The even solutions are approximated by the subsystem of even basis functions (2.10) ($i = 0, 2, 4, \ldots$) and the odd solutions by subsytem of odd functions ($i = 1, 3, 5, \ldots$). In accordance with this fact, the matrix of system (2.12) is of block-diagonal form. This enables us to find approximate solutions of even and odd types by retaining even or odd functions, respectively, in approximation (2.11).

The eigenvalues λ were determined by diagonalizing the complex matrix by means of the QR-algorithm (see [7]). In constructing the even and odd solutions we chose approximations containing as many as 16 basis functions of even or odd types. Convergence was tested by comparing the resulting values of the complex decrements in approximations containing 8 and 16 basis functions of like parity. in the range $0 \leqslant kG \leqslant 3\cdot 10^4$

the decrements of the eight lower levels of the spectrum in this approximation turned out to be practically equal.

3. The decrement spectrum. Instability. Now let us consider the results. Figures 2 and 3 show same data obtained by analyzing the decrement spectrum for a fixed value of the wavenumber $k = 1$. In accordance with the general theory [6], the property of evenness of the unperturbed profile has the effect of rendering the decrements of the normal perturbations complex $\lambda = \lambda_r + i\lambda_i$ for arbitrarily small main stream velocities (for arbitrarily small G). The real part λ_r characterizes the rate of decay (or growth) of the perturbations; the imaginary part λ_i defines their phase velocity.

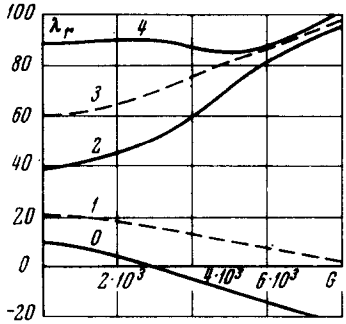

Fig. 2

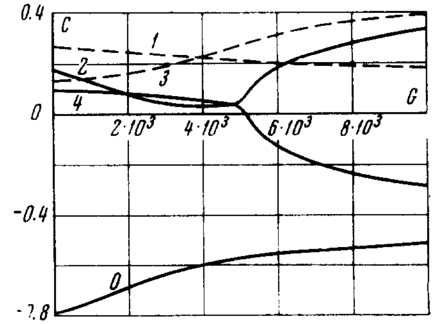

Fig. 3

Figure 2 shows the real parts of several "lower" levels of the decrement spectrum. The solid curves correspond to the even (0, 2, 4,...) perturbations; the broken curves correspond to the odd (1, 3, 5,...) ones. The levels are numbered in the order of increasing real parts λ_r for small G.

As we see from Fig. 2, the two bottom levels (the first even and the first odd levels) give rise to instability for sufficiently large G ; the real parts of the corresponding decrements (with the numbers zero and unity) become negative with increasing G. The critical values of the Grasshof number for the wavenumber $k = 1$ are $G = 2960$ and $G = 8600$.

Figure 3 shows the dimensionless phase velocities c for the same perturbations. The phase velocity unit is the maximum velocity of unperturbed motion at the channel axis (v_m). The dimensionless phase velocity determined in these units is related to the imaginary part λ_i of the decrement by the expression

$$c = \frac{60}{G}\frac{\lambda_i}{k}$$

Positive values of the phase velocity c are associated with perturbations in the form of travelling waves propagating in the positive direction of the z-axis, i.e. with perturbations deflected (swept) by the ascending central stream. Calculations indicate that the phase velocity of the same mode can have different signs for different k and G (for example, Fig. 3 shows the change in the sign of c with an increasing Grasshof number for the fourth spectral level).

The spectra λ_r (G) obtained for various wave numbers can be used to find the critical numbers G as functions of k (neutral curves along which $\lambda_r = 0$). The neutral curves for the lowest even and lowest odd levels are shown in Figs. 4 and 5. These figures also show the critical values of the phase velocities (the values of c corresponding to the points of the neutral curve).

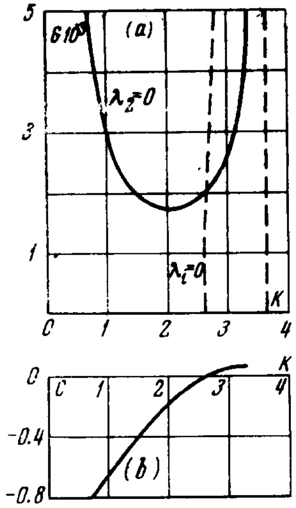

Fig. 4

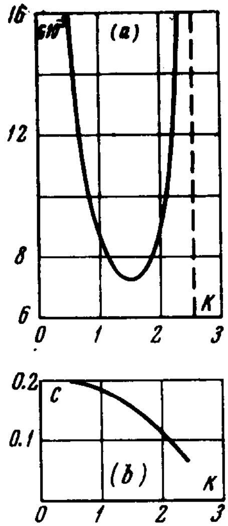

Fig. 5

Thus, calculations indicate that the steady motion under consideration becomes unstable with respect to even and odd perturbations. Perturbations of the even type are more hazardous than the odd-type perturbations: the minimum critical Grasshof number for the even perturbations is $G_m = 1720$ and occurs at the critical wavenumber $k_m = 2.05$; for odd perturbations $G_m = 7180$, $k_m = 1.57$. It is interesting to note that the phase velocity of the fundamental instability mode changes sign with changes in the parameters along the neutral curve (Fig. 4b). The long-wave neutral perturbations with $k < 2.65$ have a negative phase velocity, i. e. the perturbations are swept downward; in particular for the critical perturbation which corresponds to the minimum of the neutral curve $c_m = -0.16$. For the perturbations with $k > 2.65$, the phase velocity $c > 0$, i. e. the perturbations are swept upward. Thus, the neutral curve contains a point ($k = 2.65$) which is associated with a neutral "standing" perturbation with a zero phase velocity. This point is the intersection of the neutral curve $\lambda_r = 0$ with the curve $\lambda_i = 0$, all of whose points correspond to standing (decaying or growing) perturbations. The existence of standing perturbations, including the neutral one (at $k = 2.65$) is, of course, due to the fact that the steady flow in question is closed (the discharge equals zero) even though its profile is even.

In contrast to the fundamental instability level, the neutral critical perturbations of odd mode have a positive phase velocity for all k, i. e. they are swept upward (Fig. 5b). At the minimum of the neutral curve $c_m = 0.15$.

In order to determine the forms of the critical perturbations corresponding to the two instability levels we were obliged to find the coefficients of expansions (2.11). These coefficients are defined (to within normalization) by homogeneous system (2.12). We computed the proper vector by the Gauss method. Knowing the stream function ψ of the perturbations, we were able to construct the stream function of the overall motion, namely $\Psi = \psi_0(x) + a\psi(x, z, t)$, where ψ_0 is the stream function of the main motion (2.3).

The isolines of the stream function of the sum motion are defined by the equation $\mathrm{Re}\,\Psi = \mathrm{const}$ (Re is the real part). Figure 6 shows the form of the sum (perturbed) motion for the two instability modes. In constructing the isolines we fixed an instant $t = 0$ and determined the normalization factor a from the requirement that the maximum value of the perturbation stream function be equal to 0.1 of the maximum value of the unperturbed stream function. Both pictures of motion correpond to growing perturbations with parameters k and G close to the minima of the neutral curves. Figure 6a corresponds to the even instability mode with the parameters $k = 2$, $G = 2000$; Figure 6b corresponds to an odd mode with the parameters $k = 1.5$, $G = 8000$; the values of Re Ψ indicated in the figure have been increased by the factor 10^3 .

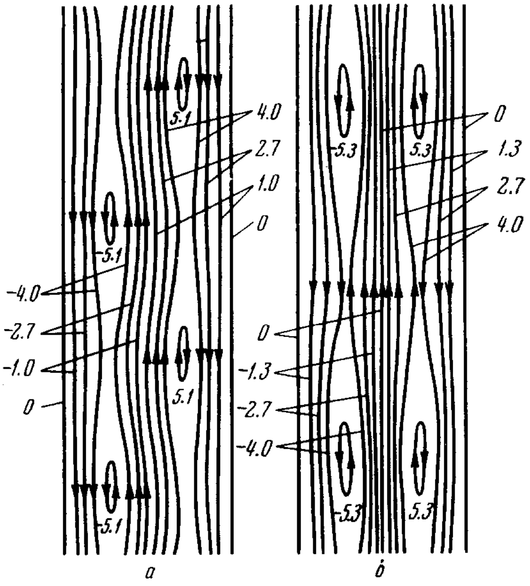

Fig. 6

The perturbed-motion structures shown in Fig. 6 provide an insight into the nature of the instability of the convective flow in question and enable us to distinguish between the two instability modes. As in the case of convective flow between planes at different temperatures, instability in this case develops in the form of a system of eddies at the boundaries between the convective counterflows. In contrast to the flow with a cubic profile, there are now two of these boundaries: one on the right and one on the left side of the channel. Accordingly, we have two chains of eddies whose relative disposition varies. The lower instability mode is associated with two eddy chains in a checkerboard configuration (Fig. 6a). The upper mode is associated with chains in a mirror-image arrangement symmetric with respect to the center of the channel (Fig. 6b). The checkerboard configuration implies more "densely packed" eddies, and is therefore preferable (since it corresponds to a smaller critical Grasshof number). Denser packing of the eddy system naturally means a smaller critical wavelength: in the checkerboard arrangement the distance between neighboring eddies (at the minimum of the neutral curve) is $3h$; in the mirror-image arrangement this distance is $4h$ (h is the half-width of the channel).

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Translated by A. Y.